Correct interaction Hamiltonian for primordial trispectrum

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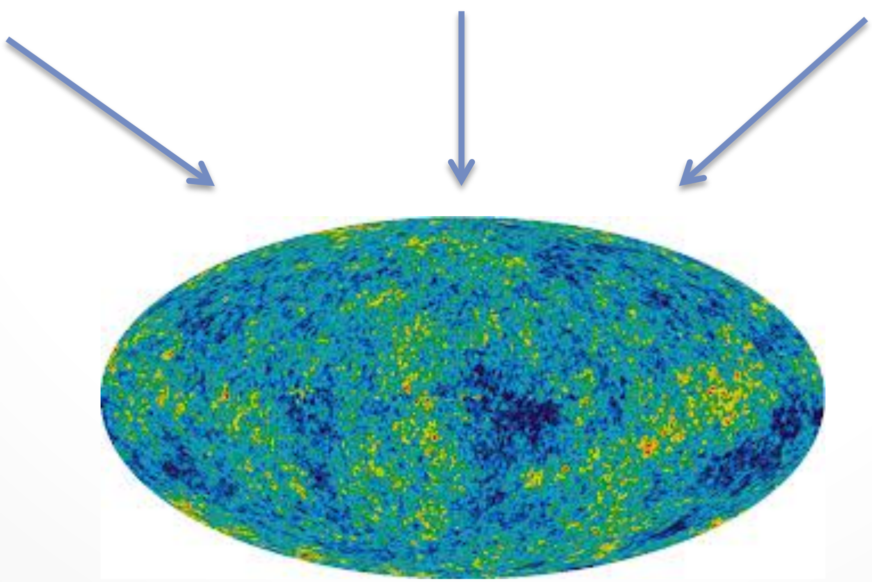
- Metric perturbation of the space-time in the inflationary universe;

  Curvature perturbation, tensor perturbation(GW)

- The 2-, 3-, 4-, … , point function of the curvature perturbation and tensor perturbation;

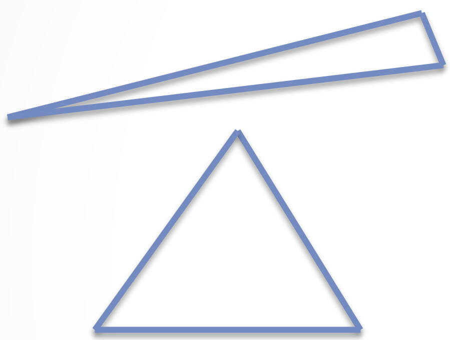
  $<\Phi\ \Phi>$, $<\gamma\ \gamma>$, $<\Phi\ \Phi\ \Phi>$, $<\Phi\ \Phi\ \gamma>$, $<\Phi\ \Phi\ \Phi\ \Phi>$, …

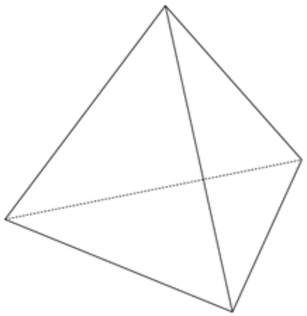
- CMB



Imprinted!

What the physical quantity can we constrain from CMB?

- The temperature perturbation of the CMB photon in the momentum space
- 

3-point correlation

Strength: f_{NL}

4-point correlation

Strength: g_{NL}

Equilateral type

No constraint to the equilateral type of g_{NL} parameter

➡ Let's calculate g_{NL}

What is needed to calculate gNL

Maldacena (2002)

$$\langle \Omega | \zeta_n(t, \mathbf{k_1}) \zeta_n(t, \mathbf{k_2}) \zeta_n(t, \mathbf{k_3}) \zeta_n(t, \mathbf{k_4}) | \Omega \rangle =$$

$$-i \int_{t_0}^{t} d\tilde{t} \langle 0 | \left[\zeta_n(t, \mathbf{k_1}) \zeta_n(t, \mathbf{k_2}) \zeta_n(t, \mathbf{k_3}) \zeta_n(t, \mathbf{k_4}), H_I(\tilde{t}) \right] | 0 \rangle$$

- gNL is generated by 4-point interaction Hamiltonian, H4
- Conventional : H4 = - L4 + …
- Derive the correct 4-point interaction Hamiltonian H4

  and

  Calculate correct gNL for a model

1. Introduction
2. The model & deriving correct Hamiltonian
3. Calculations of the 4-point function and g_{NL}
4. Conclusions & discussion

The model

- Action

  &

  bg. metric

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{pl}^2}{2}R + P(\phi, X)\right]$$

$$ds^2 = -dt^2 + a(t)^2 d\boldsymbol{x}^2 .$$

- Model parameters

$$c_s^2 \equiv \frac{P_{,X}}{\rho_{,X}} = \frac{P_{,X}}{P_{,X} + 2XP_{,XX}}$$

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{XP_{,X}}{M_{pl}^2 H^2}$$

$$\eta \equiv \frac{\dot{\epsilon}}{\epsilon H},$$

$$s \equiv \frac{\dot{c}_s}{c_s H}.$$

$$\Sigma \equiv XP_{,X} + 2X^2 P_{,XX} = \frac{M_{pl}^2 H^2 \epsilon}{c_s^2}$$

$$\lambda \equiv X^2 P_{,XX} + \frac{2}{3}X^3 P_{,XXX},$$

$$\Pi \equiv X^3 P_{,Xxx} + \frac{2}{5}X^4 P_{,XXXX} .$$

Speed of Sound

&

Slow varying parameters (SV)

Perturbations

- Metric & gauges

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

Gauge 1: $\delta\phi = 0, \quad h_{ij} = a^2 e^{2\zeta}\delta_{ij}$

ζ : curvature perturbation

Gauge 2: $\delta\phi = \varphi, \quad h_{ij} = a^2(t)\delta_{ij}$

φ : inflaton perturbation

Gauge 1 ⇔ Gauge 2 : $\zeta \simeq \zeta_n = -\frac{H}{\dot{\phi}}\varphi$ (at leadig order of SV)

- Action of ζ n at leading order of SV

Koyama et.al (2008)

$$S_{\zeta_n} = S_2 + S_3 + S_4,$$

$$S_2 = \int d\tau dx^3 \frac{M_{pl}^2 a^2 \epsilon}{c_s^2}\left(\zeta_n'^2 - a^{-2}c_s^2(\partial\zeta_n)^2\right),$$

$$S_3 = \int d\tau dx^3 \left[-aI_\epsilon \frac{\zeta'^3}{H^3} - \frac{3M_{pl}^2 a\epsilon}{c_s^4}(1-c_s^2)\zeta\zeta'^2 + \frac{a^2\epsilon}{c_s^2}(1-c_s^2)\zeta_n(\zeta_n)^2 + \cdots\right]$$

$$I_\epsilon \equiv \Sigma\left(1 - \frac{1}{c_s^2}\right) + 2\lambda,$$

$$S_4 = \int d\tau L_4 = \int d\tau dx^3 \frac{1}{H^4}\left[\left(\frac{5}{3}\Pi + \frac{1}{2}\lambda\right)\zeta_n'^4 - \left(3\lambda - H^2\epsilon\frac{1-c_s^2}{c_s^2}\right)\zeta_n'^2(\partial\zeta_n)^2 + \frac{H^2\epsilon}{4}\frac{1-c_s^2}{c_s^2}((\partial\zeta_n)^2)^2\right.$$

Deriving interaction Hamiltonian, H4 for general model

- Lagrangian & conjugate momentum

$$L = L_2(\Phi(x), \dot\Phi(x)) + L_i(\Phi(x), \dot\Phi(x))$$

$$L_2 = \int d^{(n-1)}x \left(\frac{1}{2} f_{AB}\dot\Phi_A\dot\Phi_B + \bar f_{AB}\dot\Phi_A\Phi_B - V_2(\Phi)\right)$$

$$L_i = L_3(\Phi, \dot\Phi) + L_4(\Phi, \dot\Phi)) \; .$$

$$p_A(x) \equiv \frac{\partial L}{\partial \dot\Phi_A(x)} = f_{AB}\dot\Phi_B(x) + \bar f_{AB}\Phi_B(x) + \frac{\partial L_i(\Phi, \dot\Phi)}{\partial \dot\Phi_A(x)} .$$

- Perturbative solution of $\dot\Phi_A$

$$\dot\Phi_A = \pi_A - f^{-1}_{AB}\frac{\partial L_i(\Phi,\pi)}{\partial \pi_B} + f^{-1}_{AB}\frac{\partial^2 L_3}{\partial\pi_B\partial\pi} f^{-1}\frac{\partial L_3(\Phi,\pi)}{\partial\pi} + \cdots$$

$$\pi_A(\Phi, p) = f^{-1}_{AB}(p - \bar f\Phi)_B$$

- (Interaction) Hamiltonian

$$H(\Phi, p) \equiv \int d^{(n-1)}x \; p_A \dot{\Phi}_A - L(\Phi, \dot{\Phi}(\Phi, p))$$
$$= \int d^{(n-1)}x \left(\frac{1}{2} f_{AB} \pi_A(\Phi, p) \pi_B(\Phi, p) + V_2(\Phi) \right)$$
$$- L_i(\Phi, \pi(\Phi, p)) + \frac{1}{2} \frac{\partial L_3(\Phi, \pi(\Phi, p))}{\partial \pi_A} f^{-1}_{AB} \frac{\partial L_3(\Phi, \pi(\Phi, p))}{\partial \pi_B}$$

$$H_4(\Phi, \dot{\Phi}) = \underbrace{-L_4(\Phi, \dot{\Phi})}_{\text{Conservative one}} \underbrace{- \frac{1}{2} \frac{\partial L_3}{\partial \dot{\Phi}_A} f^{-1}_{AB} \frac{\partial L_3(\Phi, \dot{\Phi})}{\partial \dot{\Phi}_B}}_{\text{New term missed ever!}}$$

Conservative one

New term missed ever!

Consistency check for H4

- E-L equation

$$\left(\frac{\partial L(\Phi,\dot{\Phi})}{\partial\dot{\Phi}_A(x)}\right)^{\cdot} - \frac{\partial L(\Phi,\dot{\Phi})}{\partial\Phi_A(x)} = 0$$

- Canonical equations

$$\dot{\Phi}_A = \frac{\partial H}{\partial p_A}$$

$$\dot{p}_A = -\frac{\partial H}{\partial \dot{\Phi}_A}$$

- Perturvative solutions of the canonical eqs.

$$\dot{\Phi}_A = \frac{\partial H}{\partial p_A}$$

$$\dot{p}_A = -\frac{\partial H}{\partial \dot{\Phi}_A}$$

$$p_A(x) = f^{-1}_{AB}\dot{\Phi}_B(x) + \bar{f}_{AB}\Phi_B + \frac{\partial L_i}{\partial \dot{\Phi}_A} + \cdots$$

$$= \frac{\partial L(\Phi, \dot{\Phi}(\Phi, \pi(\Phi, p)))}{\partial \dot{\Phi}_A(x)} + \cdots .$$

$$\dot{p}_A(x) = \frac{\partial L(\Phi, \dot{\Phi}(\Phi, \pi(\Phi, \dot{\Phi})))}{\partial \Phi_A(x)} + \cdots .$$

$$\left(\frac{\partial L(\Phi, \dot{\Phi})}{\partial \dot{\Phi}_A(x)}\right)^{\cdot} - \frac{\partial L(\Phi, \dot{\Phi})}{\partial \Phi_A(x)} + \cdots = 0$$

$$H_4(\Phi, \dot{\Phi}) = -L_4(\Phi, \dot{\Phi}) - \frac{1}{2}\frac{\partial L_3}{\partial \dot{\Phi}_A} f^{-1}_{AB} \frac{\partial L_3(\Phi, \dot{\Phi})}{\partial \dot{\Phi}_B}$$

Our formula of Hamiltonian is new, correct and simple !

4-point function and g_{NL} at leading order of SV

- New H4 for the model

$$\langle\Omega|\zeta_n(t,\mathbf{k_1})\zeta_n(t,\mathbf{k_2})\zeta_n(t,\mathbf{k_3})\zeta_n(t,\mathbf{k_4})|\Omega\rangle =$$

$$= -i\int_{t_0}^{t} d\tilde{t}\langle 0|\left[\zeta_n(t,\mathbf{k_1})\zeta_n(t,\mathbf{k_2})\zeta_n(t,\mathbf{k_3})\zeta_n(t,\mathbf{k_4}), H_I(\tilde{t})\right]|0\rangle$$

$$H_4^\epsilon = -L_4^\epsilon + \Delta H_4^\epsilon$$

$$= -\frac{1}{H^4}\int d^3x\left[\left(\frac{5}{3}\Pi+\frac{1}{2}\right)\lambda\zeta_n'^{\,4} - \left(3\lambda - H^2\epsilon\frac{1-c_s^2}{c_s^2}\right)\zeta_n'^{\,2}(\partial\zeta_n)^2 + \frac{H^2\epsilon}{4}\frac{1-c_s^2}{c_s^2}((\partial\zeta_n)^2)^2\right]$$

$$+\Delta H_4^\epsilon,$$

$$\boxed{\Delta H_4^\epsilon = -\frac{9c_s^2}{4M_{pl}^2}\int d^3x\left[\frac{I_\epsilon^2}{H^6\epsilon}\zeta_n'^4 - \frac{4aM_{pl}^2 I_\epsilon}{c_s^4 H^3}(c_s^2-1)\zeta_n\zeta_n'^3 + \frac{4M_{pl}^4 a^2\epsilon}{c_s^8}(c_s^2-1)^2\zeta_n^2\zeta_n'^2\right]}$$

$$\zeta_n(x) = \int\frac{d^3k}{(2\pi)^3}\zeta_n(\tau,\,\boldsymbol{k})\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}},$$

$$\zeta_n(\tau,\,\boldsymbol{k}) = a_{\boldsymbol{k}}u_k(\tau) + a^\dagger_{-\boldsymbol{k}}u_k^*(\tau),$$

$$u_k(\tau) = \frac{iH}{\sqrt{4M_{pl}^2 c_s\epsilon k^3}}(1+ikc_s\tau)\mathrm{e}^{-ikc_s\tau}$$

- Results at leading order of SV

Koyama et.al (2008)

$$< \zeta(\tau_f\,,\boldsymbol{k_1})\zeta(\tau_f,\boldsymbol{k_2})\zeta(\tau_f,\boldsymbol{k_3})\zeta(\tau_f,\boldsymbol{k_4}) >_c^{-L_4^\epsilon}$$
$$= (2\pi)^3\delta^{(3)}(\sum_i \boldsymbol{k}_i)\left(\frac{H^2}{4M_{pl}^2 c_s\epsilon}\right)^3 \frac{1}{\prod_i k_i^3}$$
$$\left[48(10\Pi+3\lambda)\frac{c_s^2}{M_{pl}^2H^2\epsilon}A_1 - \left(\frac{3\lambda}{M_{pl}^2H^2\epsilon}-\frac{1}{c_s^2}+1\right)A_2+\frac{1-c_s^2}{4c_s^4}A_3(\right]$$

$$A_1 = \frac{\prod_i k_i^2}{K^5},$$
$$A_2 = \frac{4k_1^2k_2^2(\boldsymbol{k}_3\cdot\boldsymbol{k}_4)}{K^3}\left(1+\frac{3(k_3+k_4)}{K}+\frac{12k_3k_4}{K^2}\right)+5\ perms.\ ,$$
$$A_3 = \frac{4(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)(\boldsymbol{k}_3\cdot\boldsymbol{k}_4)}{K}\left[1+\frac{\sum_{i<j}k_ik_j}{K^2}+\frac{3k_1k_2k_3k_4}{K^3}\left(\sum_i\frac{1}{k_i}\right)+\frac{12k_1k_2k_3k_4}{K^4}\right]+5\ perms.\ ,$$
$$K = \sum_i k_i. \tag{21}$$

$$< \zeta(\tau_f\,,\boldsymbol{k_1})\zeta(\tau_f,\boldsymbol{k_2})\zeta(\tau_f,\boldsymbol{k_3})\zeta(\tau_f,\boldsymbol{k_4}) >_c^{\Delta H_4^\epsilon}$$
$$= (2\pi)^3\delta^{(3)}(\sum_i \boldsymbol{k}_i)\left(\frac{H^2}{4M_{pl}^2 c_s\epsilon}\right)^3 \frac{1}{\prod_i k_i^3}\left[\frac{24\cdot 27 I_\epsilon^2 c_s^4}{M_{pl}^4H^4\epsilon^2}A_1 - \frac{9I_\epsilon(1-c_s^2)}{M_{pl}^2H^2\epsilon}A_4+\frac{9(1-c_s^2)^2}{2c_s^4}A_5\right] \tag{22}$$

$$A_4 = \frac{6k_1^2k_2^2k_3^2}{K^3}\left(-1-\frac{3k_4}{K}\right)+3\ perms.\ ,$$
$$A_5 = \frac{4k_1^2k_2^2}{K}\left(1+\frac{k_3+k_4}{K}+\frac{2k_3k_4}{K^2}\right)+5\ perms.$$

New contribution

gNL parameter

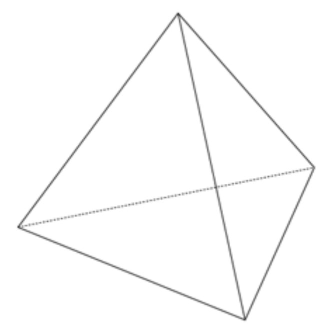
$$< \zeta_{\boldsymbol{k}_1}\zeta_{\boldsymbol{k}_2}\zeta_{\boldsymbol{k}_3}\zeta_{\boldsymbol{k}_4} >_c^{g_{NL}} = (2\pi)^3\delta^{(3)}\Big(\sum_i \boldsymbol{k}_i\Big)\cdot 6g_{NL}\left[P_\zeta^3 \frac{k_i^3}{\prod_i k_i^3} + 3\ perm.\right]$$

$$< \zeta(\tau_f, \boldsymbol{k_1})\zeta(\tau_f, \boldsymbol{k_2})\zeta(\tau_f, \boldsymbol{k_3})\zeta(\tau_f, \boldsymbol{k_4}) >_c^{\Delta H_4^\epsilon}$$

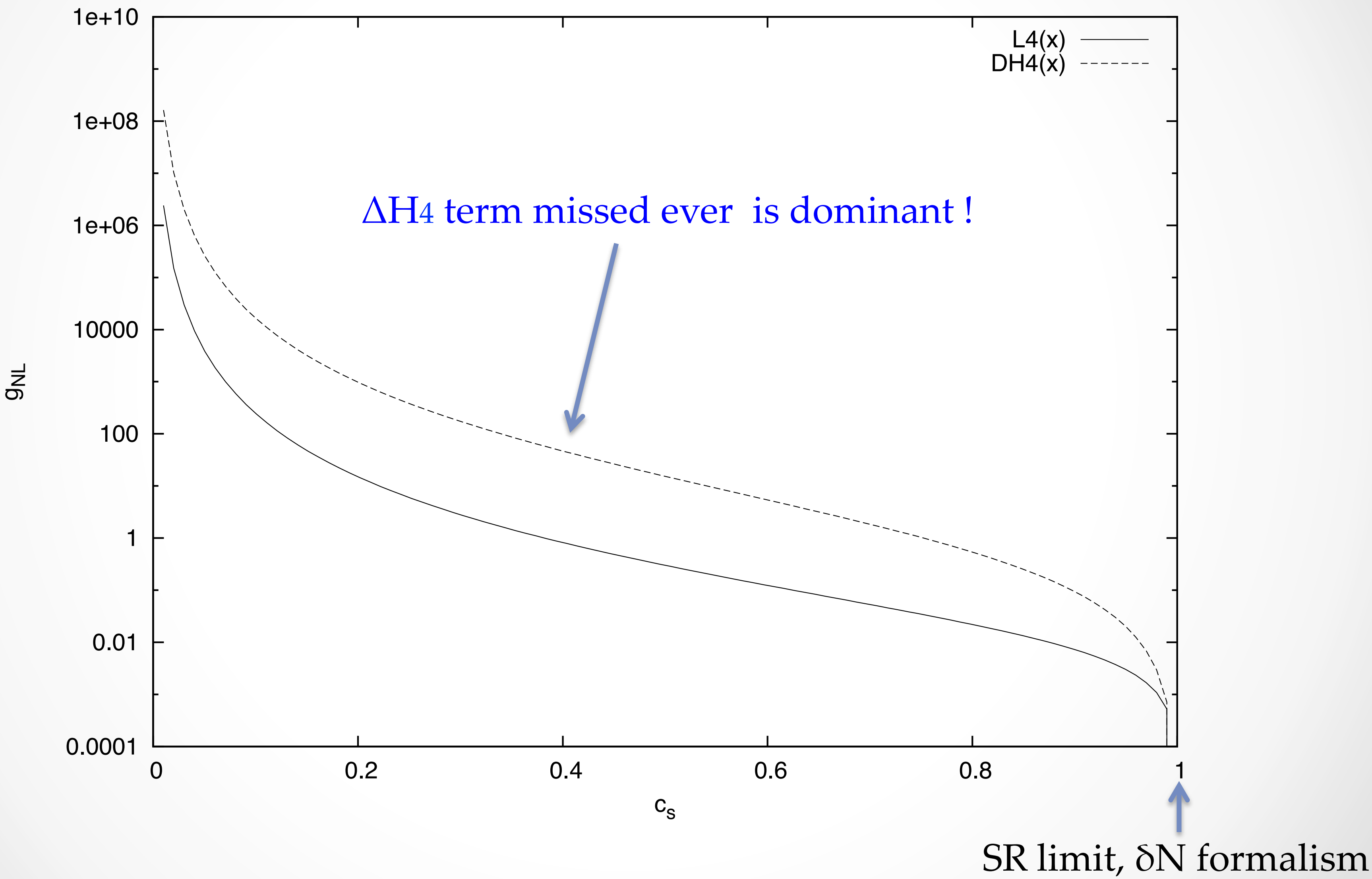
$$= (2\pi)^3\delta^{(3)}\Big(\sum_i \boldsymbol{k}_i\Big)\left(\frac{H^2}{4M_{pl}^2 c_s \epsilon}\right)^3 \frac{1}{\prod_i k_i^3}\left[\frac{24\cdot 27 I_\epsilon^2 c_s^4}{M_{pl}^4 H^4 \epsilon^2}A_1 - \frac{9I_\epsilon(1-c_s^2)}{M_{pl}^2 H^2 \epsilon}A_4 + \frac{9(1-c_s^2)^2}{2c_s^4}A_5\right] \quad (22)$$

- Equilateral type of gNL

$$k_1 = k_2 = k_3 = k_4 = \frac{K}{4}$$



$$g_{NL}^{\Delta H_4} \text{ vs. } g_{NL}^{-L_4}$$



4. Conclusions & discussion

- We derived the correct and simpler form of the 4-point interaction Hamiltonian, H_4

- The leading order result of the 4-point function of the curvature perturbation dependent to the sound speed, c_s, is suitable to the k-inflation model and consistent to the SR model and the δN formalism

- The contribution to the equilateral type of g_{NL} from ΔH_4 is about 10 ~ 100 times larger than $-L_4$

- Other shapes, More general model, Contribution from the product like the loop diagrams, …